

AFFINE QUANTUM SCHUR ALGEBRAS AND AFFINE HECKE ALGEBRAS

QIANG FU

ABSTRACT. Let F be the Schur functor from the category of finite dimensional $\mathcal{H}_\Delta(r)_\mathbb{C}$ -modules to the category of finite dimensional $\mathcal{S}_\Delta(n, r)_\mathbb{C}$ -modules, where $\mathcal{H}_\Delta(r)_\mathbb{C}$ is the extended affine Hecke algebra of type A over \mathbb{C} and $\mathcal{S}_\Delta(n, r)_\mathbb{C}$ is the affine quantum Schur algebras over \mathbb{C} . The Drinfeld polynomials associated with $F(V)$ were determined in [5, 7.6] and [7, 4.4.2] in the case of $n > r$, where V is an irreducible $\mathcal{H}_\Delta(r)_\mathbb{C}$ -module. We will generalize the result in [loc. cit.] to the case of $n \leq r$. As an application, we will classify finite dimensional irreducible $\mathcal{S}_\Delta(n, r)_\mathbb{C}$ -modules, which has been proved in [7, 4.6.8] using a different method. Furthermore we will use it to generalize [9, (6.5f)] to the affine case.

1. INTRODUCTION

It is well known that finite dimensional irreducible modules for quantum affine algebras were classified by Chari–Pressley in terms of Drinfeld polynomials (cf. [2, 3, 4, 6]). Finite dimensional irreducible modules for $\mathcal{H}_\Delta(r)_\mathbb{C}$ were classified in [13, 12], where $\mathcal{H}_\Delta(r)_\mathbb{C}$ is the extended affine Hecke algebra of type A over the complex field \mathbb{C} with a non-root of unity. The category of finite dimensional $\mathcal{H}_\Delta(r)_\mathbb{C}$ -modules and the category of finite dimensional $U_\mathbb{C}(\widehat{\mathfrak{sl}}_n)$ -modules which are of level r are related by a functor \mathcal{F} , which was defined in [5, 4.2]. Here $U_\mathbb{C}(\widehat{\mathfrak{sl}}_n)$ is quantum affine \mathfrak{sl}_n over \mathbb{C} . Chari–Pressley proved in [loc. cit.] \mathcal{F} is an equivalence of categories if $n > r$. Furthermore the Drinfeld polynomials associated with $\mathcal{F}(V)$ were determined in [loc. cit. 7.6] in the case of $n > r$, where V is an irreducible $\mathcal{H}_\Delta(r)_\mathbb{C}$ -module.

Let $U_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ be quantum affine \mathfrak{gl}_n over \mathbb{C} . In [8], finite dimensional irreducible polynomial representations of $U_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ were classified. It was proved in [7, 3.8.1] that the natural algebra homomorphism ζ_r from $U_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ to the affine quantum Schur algebra $\mathcal{S}_\Delta(n, r)_\mathbb{C}$ is surjective. Every $\mathcal{S}_\Delta(n, r)_\mathbb{C}$ -module can be regarded as a $U_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ -module via ζ_r . Let F be the Schur functor from the category of finite dimensional $\mathcal{H}_\Delta(r)_\mathbb{C}$ -modules to the category of finite dimensional $\mathcal{S}_\Delta(n, r)_\mathbb{C}$ -modules. It was proved in [7, 4.1.3 and 4.2.1] that F is an equivalence of categories in the case of $n \geq r$ and $F(V)|_{U_\mathbb{C}(\widehat{\mathfrak{sl}}_n)}$ is isomorphic to $\mathcal{F}(V)$ for any $\mathcal{H}_\Delta(r)_\mathbb{C}$ -module V . Furthermore, using [5, 7.6], the Drinfeld polynomials associated with $F(V)$ were determined in [7, 4.4.2] in the case of $n > r$, where V is an irreducible $\mathcal{H}_\Delta(r)_\mathbb{C}$ -module. We will generalize [5, 7.6] and [7, 4.4.2]

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to the case of $n \leq r$ in 4.9. Using this result, we will prove in 4.10 the classification theorem of finite dimensional irreducible $\mathcal{S}_\Delta(n, r)_\mathbb{C}$ -modules, which was established in [7, 4.6.8]. Finally, we will relate the parametrization of irreducible $\mathcal{S}_\Delta(N, r)_\mathbb{C}$ -modules, via the functor G defined in (4.10.1), to the parametrization of irreducible $\mathcal{S}_\Delta(n, r)_\mathbb{C}$ -modules in 4.11. This result is the affine version of [9, (6.5f)].

2. QUANTUM AFFINE \mathfrak{gl}_n

Let $v \in \mathbb{C}^*$ be a complex number which is not a root of unity, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Let $(c_{i,j})$ be the Cartan matrix of affine type A_{n-1} . We recall the Drinfeld's new realization of quantum affine \mathfrak{gl}_n as follows.

Definition 2.1. The *quantum loop algebra* $U_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ (or *quantum affine \mathfrak{gl}_n*) is the \mathbb{C} -algebra generated by $\mathbf{x}_{i,s}^\pm$ ($1 \leq i < n$, $s \in \mathbb{Z}$), $\mathbf{k}_i^{\pm 1}$ and $\mathbf{g}_{i,t}$ ($1 \leq i \leq n$, $t \in \mathbb{Z} \setminus \{0\}$) with the following relations:

$$(QLA1) \quad \mathbf{k}_i \mathbf{k}_i^{-1} = 1 = \mathbf{k}_i^{-1} \mathbf{k}_i, \quad [\mathbf{k}_i, \mathbf{k}_j] = 0,$$

$$(QLA2) \quad \mathbf{k}_i \mathbf{x}_{j,s}^\pm = v^{\pm(\delta_{i,j} - \delta_{i,j+1})} \mathbf{x}_{j,s}^\pm \mathbf{k}_i, \quad [\mathbf{k}_i, \mathbf{g}_{j,s}] = 0,$$

$$(QLA3) \quad [\mathbf{g}_{i,s}, \mathbf{x}_{j,t}^\pm] = \begin{cases} 0, & \text{if } i \neq j, j+1; \\ \pm v^{-js} \frac{[s]}{s} \mathbf{x}_{j,s+t}^\pm, & \text{if } i = j; \\ \mp v^{-js} \frac{[s]}{s} \mathbf{x}_{j,s+t}^\pm, & \text{if } i = j+1, \end{cases}$$

$$(QLA4) \quad [\mathbf{g}_{i,s}, \mathbf{g}_{j,t}] = 0,$$

$$(QLA5) \quad [\mathbf{x}_{i,s}^+, \mathbf{x}_{j,t}^-] = \delta_{i,j} \frac{\phi_{i,s+t}^+ - \phi_{i,s+t}^-}{v - v^{-1}},$$

$$(QLA6) \quad \mathbf{x}_{i,s}^\pm \mathbf{x}_{j,t}^\pm = \mathbf{x}_{j,t}^\pm \mathbf{x}_{i,s}^\pm, \text{ for } |i - j| > 1, \text{ and } [\mathbf{x}_{i,s+1}^\pm, \mathbf{x}_{j,t}^\pm]_{v^{\pm c_{ij}}} = -[\mathbf{x}_{j,t+1}^\pm, \mathbf{x}_{i,s}^\pm]_{v^{\pm c_{ij}}},$$

$$(QLA7) \quad [\mathbf{x}_{i,s}^\pm, [\mathbf{x}_{j,t}^\pm, \mathbf{x}_{i,p}^\pm]_v]_v = -[\mathbf{x}_{i,p}^\pm, [\mathbf{x}_{j,t}^\pm, \mathbf{x}_{i,s}^\pm]_v]_v \text{ for } |i - j| = 1,$$

where $[x, y]_a = xy - ayx$, $[s] = \frac{v^s - v^{-s}}{v - v^{-1}}$ and $\phi_{i,s}^\pm$ are defined via the generating functions in indeterminate u by

$$\Phi_i^\pm(u) := \widetilde{\mathbf{k}}_i^{\pm 1} \exp(\pm(v - v^{-1}) \sum_{m \geq 1} \mathbf{h}_{i,\pm m} u^{\pm m}) = \sum_{s \geq 0} \phi_{i,\pm s}^\pm u^{\pm s}$$

with $\widetilde{\mathbf{k}}_i = \mathbf{k}_i / \mathbf{k}_{i+1}$ ($\mathbf{k}_{n+1} = \mathbf{k}_1$) and $\mathbf{h}_{i,\pm m} = v^{\pm(i-1)m} \mathbf{g}_{i,\pm m} - v^{\pm(i+1)m} \mathbf{g}_{i+1,\pm m}$ ($1 \leq i < n$).

The algebra $U_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ has another presentation which we now describe. Let $\mathfrak{D}_{\Delta, \mathbb{C}}(n)$ be the double Ringel–Hall algebra of the cyclic quiver $\Delta(n)$. By [7, 2.3.1], the algebra $\mathfrak{D}_{\Delta, \mathbb{C}}(n)$ has the following presentation.

Lemma 2.2. *The double Ringel–Hall algebra $\mathfrak{D}_{\Delta, \mathbb{C}}(n)$ of the cyclic quiver $\Delta(n)$ is the \mathbb{C} -algebra generated by E_i , F_i , K_i , K_i^{-1} , \mathbf{z}_s^+ , \mathbf{z}_s^- , for $1 \leq i \leq n$, $s \in \mathbb{Z}^+$, and relations:*

$$(QGL1) \quad K_i K_j = K_j K_i, \quad K_i K_i^{-1} = 1;$$

$$(QGL2) \quad K_i E_j = v^{\delta_{i,j} - \delta_{i,j+1}} E_j K_i, \quad K_i F_j = v^{-\delta_{i,j} + \delta_{i,j+1}} F_j K_i;$$

$$(QGL3) \quad E_i F_j - F_j E_i = \delta_{i,j} \frac{\tilde{K}_i - \tilde{K}_i^{-1}}{v - v^{-1}}, \text{ where } \tilde{K}_i = K_i K_{i+1}^{-1};$$

$$(QGL4) \quad \sum_{a+b=1-c_{i,j}} (-1)^a \begin{bmatrix} 1 - c_{i,j} \\ a \end{bmatrix} E_i^a E_j E_i^b = 0 \text{ for } i \neq j;$$

$$(QGL5) \quad \sum_{a+b=1-c_{i,j}} (-1)^a \begin{bmatrix} 1 - c_{i,j} \\ a \end{bmatrix} F_i^a F_j F_i^b = 0 \text{ for } i \neq j;$$

$$(QGL6) \quad z_s^+ z_t^+ = z_t^+ z_s^+, z_s^- z_t^- = z_t^- z_s^-, z_s^+ z_t^- = z_t^- z_s^+;$$

$$(QGL7) \quad K_i z_s^+ = z_s^+ K_i, K_i z_s^- = z_s^- K_i;$$

$$(QGL8) \quad E_i z_s^+ = z_s^+ E_i, E_i z_s^- = z_s^- E_i, F_i z_s^- = z_s^- F_i, \text{ and } z_s^+ F_i = F_i z_s^+,$$

where $1 \leq i, j \leq n$, $s, t \in \mathbb{Z}^+$ and $\begin{bmatrix} c \\ a \end{bmatrix} = \prod_{s=1}^a \frac{v^{c-s+1} - v^{-c+s-1}}{v^s - v^{-s}}$ for $c \in \mathbb{Z}$. It is a Hopf algebra with comultiplication Δ , counit ε , and antipode σ defined by

$$\begin{aligned} \Delta(E_i) &= E_i \otimes \tilde{K}_i + 1 \otimes E_i, & \Delta(F_i) &= F_i \otimes 1 + \tilde{K}_i^{-1} \otimes F_i, \\ \Delta(K_i^{\pm 1}) &= K_i^{\pm 1} \otimes K_i^{\pm 1}, & \Delta(z_s^{\pm}) &= z_s^{\pm} \otimes 1 + 1 \otimes z_s^{\pm}; \\ \varepsilon(E_i) &= \varepsilon(F_i) = 0 = \varepsilon(z_s^{\pm}), & \varepsilon(K_i) &= 1; \\ \sigma(E_i) &= -E_i \tilde{K}_i^{-1}, & \sigma(F_i) &= -\tilde{K}_i F_i, & \sigma(K_i^{\pm 1}) &= K_i^{\mp 1}, \\ & & \text{and } \sigma(z_s^{\pm}) &= -z_s^{\pm}, \end{aligned}$$

where $1 \leq i \leq n$ and $s \in \mathbb{Z}^+$.

Let $U_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)$ be the subalgebra of $\mathfrak{D}_{\Delta\mathbb{C}}(n)$ generated by $E_i, F_i, \tilde{K}_i, \tilde{K}_i^{-1}$ for $i \in [1, n]$. Beck [1] proved that $U_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)$ is isomorphic to the subalgebra of $U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ generated by all $\mathbf{x}_{i,s}^{\pm}, \tilde{\mathbf{k}}_i^{\pm 1}$ and $\mathbf{h}_{i,t}$. The following result extends Beck's isomorphism..

Lemma 2.3 ([7, 4.4.1]). *There is a Hopf algebra isomorphism*

$$f : \mathfrak{D}_{\Delta\mathbb{C}}(n) \longrightarrow U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$$

such that

$$K_i^{\pm 1} \mapsto \mathbf{k}_i^{\pm 1}, \quad E_j \mapsto \mathbf{x}_{j,0}^+, \quad F_j \mapsto \mathbf{x}_{j,0}^- \quad (1 \leq i \leq n, 1 \leq j < n),$$

$$E_n \mapsto v \mathcal{X} \tilde{\mathbf{k}}_n, \quad F_n \mapsto v^{-1} \tilde{\mathbf{k}}_n^{-1} \mathcal{Y}, \quad z_s^{\pm} \mapsto \mp s v^{\pm s} \theta_{\pm s} \quad (s \geq 1),$$

where $\theta_{\pm s} = \mp \frac{1}{[s]_q} (\mathbf{g}_{1,\pm s} + \cdots + \mathbf{g}_{n,\pm s})$, $\mathcal{X} = [\mathbf{x}_{n-1,0}^-, [\mathbf{x}_{n-2,0}^-, \cdots, [\mathbf{x}_{2,0}^-, \mathbf{x}_{1,1}^-]_{v^{-1}} \cdots]_{v^{-1}}]_{v^{-1}}$ and $\mathcal{Y} = [\cdots [[\mathbf{x}_{1,-1}^+, \mathbf{x}_{2,0}^+]_v, \mathbf{x}_{3,0}^+]_v, \cdots, \mathbf{x}_{n-1,0}^+]_v$.

We now review the classification theorem of finite dimensional irreducible polynomial $U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ -modules. We first need to introduce the elements $\mathcal{Q}_{i,s} \in U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$, which will be used to define pseudo-highest weight modules. For $1 \leq i \leq n$ and $s \in \mathbb{Z}$, define the elements $\mathcal{Q}_{i,s} \in U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ through the generating functions

$$\mathcal{Q}_i^{\pm}(u) := \exp \left(- \sum_{t \geq 1} \frac{1}{[t]} g_{i,\pm t} (vu)^{\pm t} \right) = \sum_{s \geq 0} \mathcal{Q}_{i,\pm s} u^{\pm s} \in U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)[[u, u^{-1}]].$$

For a representation V of $U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$, a nonzero vector $w \in V$ is called a *pseudo-highest weight vector* if there exist some $Q_{i,s} \in \mathbb{C}$ such that

$$(2.3.1) \quad \mathbf{x}_{j,s}^+ w = 0, \quad \mathcal{Q}_{i,s} w = Q_{i,s} w, \quad \mathbf{k}_i w = v^{\lambda_i} w$$

for all $1 \leq i \leq n$ and $1 \leq j \leq n-1$ and $s \in \mathbb{Z}$. The module V is called a *pseudo-highest weight module* if $V = U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)w$ for some pseudo-highest weight vector w . We also write the short form $\mathcal{Q}_i^{\pm}(u)w = Q_i^{\pm}(u)w$ for the relations $\mathcal{Q}_{i,s} w = Q_{i,s} w$ ($s \in \mathbb{Z}$), where

$$Q_i^{\pm}(u) = \sum_{s \geq 0} Q_{i,\pm s} u^{\pm s}.$$

Let V be a finite dimensional polynomial representation of $U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ of type 1. Then $V = \bigoplus_{\lambda \in \mathbb{N}^n} V_{\lambda}$, where

$$V_{\lambda} = \{x \in V \mid \mathbf{k}_j x = v^{\lambda_j} x, 1 \leq j \leq n\},$$

and, since all $\mathcal{Q}_{i,s}$ commute with the \mathbf{k}_j , each V_{λ} is a direct sum of generalized eigenspaces of the form

$$(2.3.2) \quad V_{\lambda,\gamma} = \{x \in V_{\lambda} \mid (\mathcal{Q}_{i,s} - \gamma_{i,s})^p x = 0 \text{ for some } p (1 \leq i \leq n, s \in \mathbb{Z})\},$$

where $\gamma = (\gamma_{i,s})$ with $\gamma_{i,s} \in \mathbb{C}$. Let $\Gamma_i^{\pm}(u) = \sum_{s \geq 0} \gamma_{i,\pm s} u^{\pm s}$.

A finite dimensional $U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ -module V is called a *polynomial representation* if the restriction of V to $U_{\mathbb{C}}(\mathfrak{gl}_n)$ is a polynomial representation of type 1 and, for every weight $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n$ of V , the formal power series $\Gamma_i^{\pm}(u)$ associated to the eigenvalues $(\gamma_{i,s})_{s \in \mathbb{Z}}$ defining the generalized eigenspaces $V_{\lambda,\gamma}$ as given in (2.3.2), are polynomials in u^{\pm} of degree λ_i so that the zeroes of the functions $\Gamma_i^+(u)$ and $\Gamma_i^-(u)$ are the same.

Following [8], an n -tuple of polynomials $\mathbf{Q} = (Q_1(u), \dots, Q_n(u))$ with constant terms 1 is called *dominant* if, for each $1 \leq i \leq n-1$, the ratio $Q_i(v^{i-1}u)/Q_{i+1}(v^{i+1}u)$ is a polynomial. Let $\mathcal{Q}(n)$ be the set of dominant n -tuples of polynomials.

For $g(u) = \prod_{1 \leq i \leq m} (1 - a_i u) \in \mathbb{C}[u]$ with constant term 1 and $a_i \in \mathbb{C}^*$, define

$$(2.3.3) \quad g^{\pm}(u) = \prod_{1 \leq i \leq m} (1 - a_i^{\pm 1} u^{\pm 1}).$$

For $\mathbf{Q} = (Q_1(u), \dots, Q_n(u)) \in \mathcal{Q}(n)$, define $Q_{i,s} \in \mathbb{C}$, for $1 \leq i \leq n$ and $s \in \mathbb{Z}$, by the following formula

$$Q_i^{\pm}(u) = \sum_{s \geq 0} Q_{i,\pm s} u^{\pm s},$$

where $Q_i^{\pm}(u)$ is defined using (2.3.3). Let $I(\mathbf{Q})$ be the left ideal of $U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ generated by $\mathbf{x}_{j,s}^+$, $\mathcal{Q}_{i,s} - Q_{i,s}$, and $\mathbf{k}_i - v^{\lambda_i}$, for $1 \leq j \leq n-1$, $1 \leq i \leq n$ and $s \in \mathbb{Z}$, where $\lambda_i = \deg Q_i(u)$, and define

$$M(\mathbf{Q}) = U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)/I(\mathbf{Q}).$$

Then $M(\mathbf{Q})$ has a unique irreducible quotient, denoted by $L(\mathbf{Q})$. The polynomials $Q_i(u)$ are called *Drinfeld polynomials* associated with $L(\mathbf{Q})$.

Theorem 2.4 ([8]). *The $U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ -modules $L(\mathbf{Q})$ with $\mathbf{Q} \in \mathcal{Q}(n)$ are all nonisomorphic finite dimensional irreducible polynomial representations of $U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$.*

If $\mathbf{Q}, \mathbf{Q}' \in \mathcal{Q}(n)$ is such that $Q_j(v^{j-1}u)/Q_{j+1}(v^{j+1}u) = Q'_j(v^{j-1}u)/Q'_{j+1}(v^{j+1}u)$ and $\deg Q_j(u) - \deg Q_{j+1}(u) = \deg Q'_j(u) - \deg Q'_{j+1}(u)$ for $1 \leq j \leq n-1$, then by [7, 4.7.1 and 4.7.2], we have $L(\mathbf{Q})|_{U_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)} \cong L(\mathbf{Q}')|_{U_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)}$. Thus we may denote $L(\mathbf{Q})|_{U_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)}$ by $\bar{L}(\mathbf{P})$, where $\mathbf{P} = (P_1(u), \dots, P_{n-1}(u))$ with $P_j(u) = Q_j(v^{j-1}u)/Q_{j+1}(v^{j+1}u)$.

Let $\mathcal{P}(n)$ be the set of $(n-1)$ -tuples of polynomials with constant terms 1. The following result is due to Chari–Pressley (cf. [2, 3, 4]).

Theorem 2.5. *The modules $\bar{L}(\mathbf{P})$ with $\mathbf{P} \in \mathcal{P}(n)$ are all nonisomorphic finite dimensional irreducible $U_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)$ -modules of type 1.*

3. AFFINE QUANTUM SCHUR ALGEBRAS

In this section we collect some facts about extended affine Hecke algebras and affine quantum Schur algebras, which will be used in §4. The extended affine Hecke algebra $\mathcal{H}_{\Delta}(r)_{\mathbb{C}}$ is defined to be the algebra generated by

$$T_i, \quad X_j^{\pm 1} (1 \leq i \leq r-1, 1 \leq j \leq r),$$

and relations

$$\begin{aligned} (T_i + 1)(T_i - v^2) &= 0, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \quad (|i - j| > 1), \\ X_i X_i^{-1} &= 1 = X_i^{-1} X_i, \quad X_i X_j = X_j X_i, \\ T_i X_i T_i &= v^2 X_{i+1}, \quad X_j T_i = T_i X_j \quad (j \neq i, i+1). \end{aligned}$$

Let \mathfrak{S}_r be the symmetric group with generators $s_i := (i, i+1)$ for $1 \leq i \leq r-1$. Let $I(n, r) = \{(i_1, \dots, i_r) \in \mathbb{Z}^r \mid 1 \leq i_k \leq n, \forall k\}$. The symmetric group \mathfrak{S}_r acts on the set $I(n, r)$ by place permutation:

$$\mathbf{i}w = (i_{w(k)})_{k \in \mathbb{Z}}, \quad \text{for } \mathbf{i} \in I(n, r) \text{ and } w \in \mathfrak{S}_r.$$

Let $\Omega_{\mathbb{C}}$ be a vector space over \mathbb{C} with basis $\{\omega_i \mid i \in \mathbb{Z}\}$. For $\mathbf{i} = (i_1, \dots, i_r) \in \mathbb{Z}^r$, write

$$\omega_{\mathbf{i}} = \omega_{i_1} \otimes \omega_{i_2} \otimes \cdots \otimes \omega_{i_r} = \omega_{i_1} \omega_{i_2} \cdots \omega_{i_r} \in \Omega_{\mathbb{C}}^{\otimes r}.$$

The tensor space $\Omega_{\mathbb{C}}^{\otimes r}$ admits a right $\mathcal{H}_{\Delta}(r)_{\mathbb{C}}$ -module structure defined by

$$\begin{cases} \omega_{\mathbf{i}} \cdot X_t^{-1} = \omega_{i_1} \cdots \omega_{i_{t-1}} \omega_{i_t+n} \omega_{i_{t+1}} \cdots \omega_{i_r}, & \text{for all } \mathbf{i} \in \mathbb{Z}^r; \\ \omega_{\mathbf{i}} \cdot T_k = \begin{cases} v^2 \omega_{\mathbf{i}}, & \text{if } i_k = i_{k+1}; \\ v \omega_{\mathbf{i}s_k}, & \text{if } i_k < i_{k+1}; \\ v \omega_{\mathbf{i}s_k} + (v^2 - 1) \omega_{\mathbf{i}}, & \text{if } i_{k+1} < i_k, \end{cases} & \text{for all } \mathbf{i} \in I(n, r), \end{cases}$$

where $1 \leq k \leq r-1$ and $1 \leq t \leq r$.

The algebra

$$\mathcal{S}_{\Delta}(n, r)_{\mathbb{C}} := \text{End}_{\mathcal{H}_{\Delta}(r)_{\mathbb{C}}}(\mathcal{T}_{\Delta}(n, r))$$

is called an affine q -Schur algebra, where $\mathcal{T}_{\Delta}(n, r) = \Omega_{\mathbb{C}}^{\otimes r}$. Let $\Omega_{n, \mathbb{C}}$ be the subspace of $\Omega_{\mathbb{C}}$ spanned by ω_i with $1 \leq i \leq n$ and $\mathcal{H}(r)_{\mathbb{C}}$ be the subalgebra of $\mathcal{H}_{\Delta}(r)_{\mathbb{C}}$ generated by T_k for $1 \leq k \leq r-1$. Then the algebra $\mathcal{S}(n, r)_{\mathbb{C}} := \text{End}_{\mathcal{H}(r)_{\mathbb{C}}}(\mathcal{T}(n, r))$ is called a q -Schur algebra, where $\mathcal{T}(n, r) = \Omega_{n, \mathbb{C}}^{\otimes r}$.

The algebras $U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ and $\mathcal{S}_{\Delta}(n, r)_{\mathbb{C}}$ are related by an algebra homomorphism ζ_r , which we now describe. For $i \in \mathbb{Z}$, let \bar{i} denote the integer modulo n . The complex vector space $\Omega_{\mathbb{C}}$ is a natural $\mathfrak{D}_{\Delta, \mathbb{C}}(n)$ -module with the action

$$(3.0.1) \quad \begin{aligned} E_i \cdot \omega_s &= \delta_{\bar{i}+1, \bar{s}} \omega_{s-1}, & F_i \cdot \omega_s &= \delta_{\bar{i}, \bar{s}} \omega_{s+1}, & K_i^{\pm 1} \cdot \omega_s &= v^{\pm \delta_{\bar{i}, \bar{s}}} \omega_s, \\ \mathbf{z}_t^+ \cdot \omega_s &= \omega_{s-tn}, & \text{and } \mathbf{z}_t^- \cdot \omega_s &= \omega_{s+tn}. \end{aligned}$$

The Hopf algebra structure induces a $\mathfrak{D}_{\Delta, \mathbb{C}}(n)$ -module $\Omega_{\mathbb{C}}^{\otimes r}$. By [7, 3.5.5], the actions of $\mathfrak{D}_{\Delta, \mathbb{C}}(n)$ and $\mathcal{H}_{\Delta}(r)_{\mathbb{C}}$ on $\Omega_{\mathbb{C}}^{\otimes r}$ are commute. We will identify $\mathfrak{D}_{\Delta, \mathbb{C}}(n)$ and $U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ via the algebra isomorphism f defined in 2.3. Consequently, there is an algebra homomorphism

$$\zeta_r : U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n) = \mathfrak{D}_{\Delta, \mathbb{C}}(n) \longrightarrow \mathcal{S}_{\Delta}(n, r)_{\mathbb{C}}.$$

It is proved in [7, 3.8.1] that ζ_r is surjective. Let $U_{\mathbb{C}}(\mathfrak{gl}_n)$ be the subalgebra of $\mathfrak{D}_{\Delta, \mathbb{C}}(n)$ generated by E_i, F_i, K_j, K_j^{-1} for $1 \leq i \leq n-1$ and $1 \leq j \leq n$. The restriction of ζ_r to $U_{\mathbb{C}}(\mathfrak{gl}_n)$ induces a surjective algebra homomorphism $\zeta_r : U_{\mathbb{C}}(\mathfrak{gl}_n) \longrightarrow \mathcal{S}(n, r)_{\mathbb{C}}$ (cf. [10]). Every $\mathcal{S}_{\Delta}(n, r)_{\mathbb{C}}$ -module (resp., $\mathcal{S}(n, r)_{\mathbb{C}}$ -module) will be inflated into a $U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ -module (resp., $U_{\mathbb{C}}(\mathfrak{gl}_n)$ -module) via ζ_r .

The following easy lemma relates $\Omega_{\mathbb{C}}^{\otimes r}$ with $\Omega_{n, \mathbb{C}}^{\otimes r}$.

Lemma 3.1 ([7, 4.1.1]). *There is a $U_{\mathbb{C}}(\mathfrak{gl}_n)$ - $\mathcal{H}_{\Delta}(r)_{\mathbb{C}}$ -bimodule isomorphism*

$$\Omega_{n, \mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} \mathcal{H}_{\Delta}(r)_{\mathbb{C}} \xrightarrow{\sim} \Omega_{\mathbb{C}}^{\otimes r}, \quad x \otimes h \longmapsto xh.$$

The irreducible $\mathcal{H}_{\Delta}(r)_{\mathbb{C}}$ -modules were classified in [13, 12], which we now describe. For $\mathbf{a} = (a_1, \dots, a_r) \in (\mathbb{C}^*)^r$, let $M_{\mathbf{a}} = \mathcal{H}_{\Delta}(r)_{\mathbb{C}}/J_{\mathbf{a}}$, where $J_{\mathbf{a}}$ is the left ideal of $\mathcal{H}_{\Delta}(r)_{\mathbb{C}}$ generated by $X_j - a_j$ for $1 \leq j \leq r$.

A *segment* \mathbf{s} with center $a \in \mathbb{C}^*$ is by definition an ordered sequence

$$\mathbf{s} = (av^{-k+1}, av^{-k+3}, \dots, av^{k-1}) \in (\mathbb{C}^*)^k.$$

Here k is called the length of the segment, denoted by $|\mathbf{s}|$. If $\mathbf{s} = \{\mathbf{s}_1, \dots, \mathbf{s}_p\}$ is an unordered collection of segments, define $\wp(\mathbf{s})$ to be the partition associated with the sequence $(|\mathbf{s}_1|, \dots, |\mathbf{s}_p|)$. That is, $\wp(\mathbf{s}) = (|\mathbf{s}_{i_1}|, \dots, |\mathbf{s}_{i_p}|)$ with $|\mathbf{s}_{i_1}| \geq \dots \geq |\mathbf{s}_{i_p}|$, where $|\mathbf{s}_{i_1}|, \dots, |\mathbf{s}_{i_p}|$ is a permutation of $|\mathbf{s}_1|, \dots, |\mathbf{s}_p|$. We also call $|\mathbf{s}| := |\wp(\mathbf{s})|$ the length of \mathbf{s} .

Let \mathcal{S}_r be the set of unordered collections of segments \mathbf{s} with $|\mathbf{s}| = r$. Then $\mathcal{S}_r = \cup_{\mu \in \Lambda^+(r)} \mathcal{S}_{r,\mu}$, where $\mathcal{S}_{r,\mu} = \{\mathbf{s} \in \mathcal{S}_r \mid \wp(\mathbf{s}) = \mu\}$ and $\Lambda^+(r)$ is the set of partitions of r .

If $w = s_{i_1}s_{i_2} \cdots s_{i_m}$ is reduced let $T_w = T_{i_1}T_{i_2} \cdots T_{i_m}$. For $p \geq 1$ let

$$(3.1.1) \quad \Lambda(p, r) = \{\mu \in \mathbb{N}^p \mid \sum_{1 \leq i \leq p} \mu_i = r\}$$

For $\mu \in \Lambda(p, r)$ let \mathfrak{S}_μ be the corresponding standard Young subgroup of the symmetric group \mathfrak{S}_r , and let $\mathcal{D}_\mu = \{d \in \mathfrak{S}_r \mid \ell(wd) = \ell(w) + \ell(d) \text{ for } w \in \mathfrak{S}_\mu\}$. For $\mu \in \Lambda(p, r)$ let

$$(3.1.2) \quad \mathcal{I}_\mu = \mathcal{H}(r)_{\mathbb{C}} y_\mu,$$

where $y_\mu = \sum_{w \in \mathfrak{S}_\mu} (-v^2)^{-\ell(w)} T_w \in \mathcal{H}(r)_{\mathbb{C}}$. For $\mathbf{s} = \{\mathbf{s}_1, \dots, \mathbf{s}_p\} \in \mathcal{S}_{r,\mu}$, let $\mathbf{a}(\mathbf{s}) = (\mathbf{s}_1, \dots, \mathbf{s}_p) \in (\mathbb{C}^*)^r$ be the r -tuple obtained by juxtaposing the segments in \mathbf{s} . Let $\iota : \mathcal{H}(r)_{\mathbb{C}} \rightarrow M_{\mathbf{a}(\mathbf{s})}$ be the natural $\mathcal{H}(r)_{\mathbb{C}}$ -module isomorphism defined by sending h to \bar{h} . Let

$$\bar{\mathcal{I}}_\mu = \iota(\mathcal{I}_\mu) = \mathcal{H}(r)_{\mathbb{C}} \bar{y}_\mu = \mathcal{H}_\Delta(r)_{\mathbb{C}} \bar{y}_\mu.$$

Then,

$$(3.1.3) \quad \mathcal{H}(r)_{\mathbb{C}} y_\mu \cong E_\mu \oplus \left(\bigoplus_{\nu \vdash r, \nu \triangleright \mu} m_{\nu,\mu} E_\nu \right),$$

where E_ν is the left cell module defined by the Kazhdan–Lusztig’s C-basis [11] associated with the left cell containing $w_{0,\nu}$.

Let $V_{\mathbf{s}}$ be the unique composition factor of the $\mathcal{H}_\Delta(r)_{\mathbb{C}}$ -module $\mathcal{H}_\Delta(r)_{\mathbb{C}} \bar{y}_\mu$ such that the multiplicity of E_μ in $V_{\mathbf{s}}$ as an $\mathcal{H}(r)_{\mathbb{C}}$ -module is nonzero.

The following classification theorem is due to Zelevinsky [13] and Rogawski [12].

Theorem 3.2. *The modules $V_{\mathbf{s}}$ with $\mathbf{s} \in \mathcal{S}_r$ are all nonisomorphic finite dimensional irreducible $\mathcal{H}_\Delta(r)_{\mathbb{C}}$ -modules.*

Let $\mathcal{S}_\Delta(n, r)_{\mathbb{C}\text{-mod}}$ (resp., $\mathcal{H}_\Delta(r)_{\mathbb{C}\text{-mod}}$) be the category of finite dimensional $\mathcal{S}_\Delta(n, r)_{\mathbb{C}}$ -modules (resp., $\mathcal{H}_\Delta(r)_{\mathbb{C}}$ -modules). The categories $\mathcal{S}_\Delta(n, r)_{\mathbb{C}\text{-mod}}$ and $\mathcal{H}_\Delta(r)_{\mathbb{C}\text{-mod}}$ are related by the Schur functor F , which we now define. Using the $\mathcal{S}_\Delta(n, r)_{\mathbb{C}}\text{--}\mathcal{H}_\Delta(r)_{\mathbb{C}}$ -bimodule $\Omega_{\mathbb{C}}^{\otimes r}$, we define a functor

$$(3.2.1) \quad F = F_{n,r} : \mathcal{H}_\Delta(r)_{\mathbb{C}\text{-mod}} \longrightarrow \mathcal{S}_\Delta(n, r)_{\mathbb{C}\text{-mod}}, \quad V \longmapsto \Omega_{\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}_\Delta(r)_{\mathbb{C}}} V.$$

Let

$$\mathcal{S}_r^{(n)} = \{\mathbf{s} = \{\mathbf{s}_1, \dots, \mathbf{s}_p\} \in \mathcal{S}_r, p \geq 1, |\mathbf{s}_i| \leq n, \forall i\}.$$

The following classification theorem is given in [7, 4.3.4 and 4.5.3].

Lemma 3.3. *For $\mathbf{s} \in \mathcal{S}_r$ we have $F(V_{\mathbf{s}}) \neq 0$ if and only if $\mathbf{s} \in \mathcal{S}_r^{(n)}$. Furthermore, the set*

$$\{F(V_{\mathbf{s}}) \mid \mathbf{s} \in \mathcal{S}_r^{(n)}\}$$

is a complete set of nonisomorphic finite dimensional irreducible $\mathcal{S}_{\Delta}(n, r)_{\mathbb{C}}$ -modules.

The following result, which will be used in 4.9, is taken from [5, 7.6] and [7, 4.4.2 and 4.6.5].

Lemma 3.4. *Assume $n \geq r$. Let $\mathbf{s} = (av^{-r+1}, av^{-r+3}, \dots, av^{r-1})$ be a single segment and $\mu = \wp(\mathbf{s}) = (r)$. Then $V_{\mathbf{s}} = \bar{\mathcal{I}}_{\mu}$ and $F(V_{\mathbf{s}}) \cong L(\mathbf{Q})$, where $\mathbf{Q} = (Q_1(u), \dots, Q_n(u))$ with $Q_n(u) = (1 - av^{-n+1}u)^{\delta_{n,r}}$ and $\frac{Q_i(uv^{i-1})}{Q_{i+1}(uv^{i+1})} = (1 - au)^{\delta_{i,r}}$ for $1 \leq i \leq n-1$.*

4. IDENTIFICATION OF IRREDUCIBLE $\mathcal{S}_{\Delta}(n, r)_{\mathbb{C}}$ -MODULES

In this section we will prove that $F(\bar{\mathcal{I}}_{\wp(\mathbf{s})})$ is isomorphic to the tensor product of irreducible $\mathcal{S}_{\Delta}(n, r)_{\mathbb{C}}$ -modules for $\mathbf{s} \in \mathcal{S}_r^{(n)}$ and $F(\bar{\mathcal{I}}_{\wp(\mathbf{s})}) = 0$ for $\mathbf{s} \notin \mathcal{S}_r^{(n)}$ in 4.6. Using this result, we will relate the parametrization of irreducible $\mathcal{H}_{\Delta}(r)_{\mathbb{C}}$ -modules, via the functor F defined in (3.2.1), to the parametrization of finite dimensional irreducible polynomial representations of $U_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ in 4.9. As applications, we will classify finite dimensional irreducible $\mathcal{S}_{\Delta}(n, r)_{\mathbb{C}}$ -modules in 4.10, and generalize [9, (6.5f)] to the affine case.

To compute $F(\bar{\mathcal{I}}_{\wp(\mathbf{s})})$, we need a result of Rogawski [12, 4.3], which we now describe. For $1 \leq j \leq p$, let $\mathcal{H}_{\mu,j}$ be the subalgebra of $\mathcal{H}(r)_{\mathbb{C}}$ generated by T_i with $s_i \in \mathfrak{S}_{\mu(j)}$, where

$$\mu^{(j)} = (1^{\mu_{[1,j-1]}}, \mu_j, 1^{r-\mu_{[1,j]}},$$

and $\mu_{[1,j]} = \mu_1 + \mu_2 + \dots + \mu_j$. Since $\mathcal{H}_{\mu,j} \cong \mathcal{H}(\mu_j)_{\mathbb{C}}$ for $1 \leq j \leq p$ and $\Omega_{n,\mathbb{C}}^{\otimes \mu_j}$ is a right $\mathcal{H}(\mu_j)_{\mathbb{C}}$ -module, $\Omega_{n,\mathbb{C}}^{\otimes \mu_j}$ can be also regarded as a right $\mathcal{H}_{\mu,j}$ -module.

Recall the notation \mathcal{I}_{μ} defined in (3.1.2). For $\mu \in \Lambda(p, r)$ and $1 \leq j \leq p$ let

$$\mathcal{J}_{\mu} = \bigcap_{\substack{s_i \in \mathfrak{S}_{\mu} \\ 1 \leq i \leq r-1}} \mathcal{H}(r)_{\mathbb{C}} C_i, \quad \mathcal{J}_{\mu,j} = \bigcap_{\substack{s_i \in \mathfrak{S}_{\mu^{(j)}} \\ 1 \leq i \leq r-1}} \mathcal{H}_{\mu,j} C_i \quad \text{and} \quad \mathcal{I}_{\mu,j} = \mathcal{H}_{\mu,j} y_{\mu(j)}.$$

where $C_i = v^{-1}T_i - v$ and $y_{\mu(j)} = \sum_{w \in \mathfrak{S}_{\mu(j)}} (-v^2)^{-\ell(w)} T_w$. By [12, 4.3] we have the following result.

Lemma 4.1. *We have $\mathcal{I}_{\mu} = \mathcal{J}_{\mu}$, $\mathcal{I}_{\mu,j} = \mathcal{J}_{\mu,j}$ for $\mu \in \Lambda(p, r)$ and $1 \leq j \leq p$.*

Lemma 4.2. *Assume I is a left ideal of $\mathcal{H}(r)_{\mathbb{C}}$. Then $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} I \cong \Omega_{n,\mathbb{C}}^{\otimes r} I$.*

Proof. Since $\mathcal{H}(r)_{\mathbb{C}}$ is semisimple, there exist a left ideal J of $\mathcal{H}(r)_{\mathbb{C}}$ such that $\mathcal{H}(r)_{\mathbb{C}} = I \oplus J$. Then $\Omega_{n,\mathbb{C}}^{\otimes r} \cong \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} \mathcal{H}(r)_{\mathbb{C}} \cong \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} I \oplus \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} J$. Thus the natural linear map $f : \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} I \rightarrow \Omega_{n,\mathbb{C}}^{\otimes r}$ defined by sending $w \otimes h$ to wh is injective. Consequently, $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} I \cong \text{Im}(f) = \Omega_{n,\mathbb{C}}^{\otimes r} I$. \square

By 3.1, 4.1 and 4.2 we conclude that $F(\bar{\mathcal{I}}_{\mu}) \cong \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} \bar{\mathcal{J}}_{\mu} \cong \Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{J}_{\mu}$, where $\mu = \wp(\mathbf{s})$ for some $\mathbf{s} \in \mathcal{S}_r$. We now compute $\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{J}_{\mu}$.

Lemma 4.3. *For $\mu \in \Lambda(p, r)$, we have*

$$\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{J}_{\mu} = \Omega_{n,\mathbb{C}}^{\otimes \mu_1} \mathcal{J}_{\mu,1} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_p} \mathcal{J}_{\mu,p}.$$

Proof. Since $\mathcal{J}_{\mu} = \cap_{1 \leq j \leq p} \mathcal{J}_{\mu(j)}$ we have $\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{J}_{\mu} \subseteq \cap_{1 \leq j \leq p} (\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{J}_{\mu(j)})$. Furthermore by 4.1 we have $\mathcal{J}_{\mu(j)} = \mathcal{I}_{\mu(j)} = \mathcal{X}_{\mu,j} \mathcal{I}_{\mu,j} = \mathcal{X}_{\mu,j} \mathcal{J}_{\mu,j}$ where $\mathcal{X}_{\mu,j} = \text{span}\{T_w \mid w \in \mathcal{D}_{\mu(j)}^{-1}\}$. This implies that

$$\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{J}_{\mu(j)} = \Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{J}_{\mu,j} = \Omega_{n,\mathbb{C}}^{\mu_1} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\mu_j-1} \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_j} \mathcal{J}_{\mu,j} \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_{j+1}} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_p}$$

for $1 \leq j \leq p$. Thus,

$$\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{J}_{\mu} \subseteq \bigcap_{1 \leq j \leq p} (\Omega_{n,\mathbb{C}}^{\mu_1} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\mu_j-1} \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_j} \mathcal{J}_{\mu,j} \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_{j+1}} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_p}) = \Omega_{n,\mathbb{C}}^{\otimes \mu_1} \mathcal{J}_{\mu,1} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_p} \mathcal{J}_{\mu,p}.$$

On the other hand, we assume $w_1 h_1 \otimes \cdots \otimes w_p h_p \in \Omega_{n,\mathbb{C}}^{\otimes \mu_1} \mathcal{J}_{\mu,1} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_p} \mathcal{J}_{\mu,p}$, where $w_j \in \Omega_{n,\mathbb{C}}^{\otimes \mu_j}$ and $h_j \in \mathcal{J}_{\mu,j}$. Since $h_k h_l = h_l h_k$ for any k, l and $h_j \in \mathcal{J}_{\mu,j}$, we have $h_1 h_2 \cdots h_p = (h_1 \cdots h_{j-1} h_{j+1} \cdots h_p) h_j \in \mathcal{H}(r)_{\mathbb{C}} \mathcal{J}_{\mu,j} \subseteq \mathcal{H}(r)_{\mathbb{C}} C_i$ for $1 \leq i \leq r-1$, $1 \leq j \leq p$ with $s_i \in \mathfrak{S}_{\mu(j)}$. This implies that $h_1 h_2 \cdots h_p \in \mathcal{J}_{\mu}$. It follows that $w_1 h_1 \otimes \cdots \otimes w_p h_p = (w_1 \otimes \cdots \otimes w_p) h_1 \cdots h_p \in \Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{J}_{\mu}$. The assertion follows. \square

For $\mu \in \Lambda(p, r)$ and $1 \leq j \leq p$, let $\tilde{\mathcal{H}}_{\mu,j}$ be the subalgebra of $\mathcal{H}_{\Delta}(r)_{\mathbb{C}}$ generated by T_i and $X_{\mu_{[1,j-1]}+1}, \dots, X_{\mu_{[1,j]}}$ with $s_i \in \mathfrak{S}_{\mu(j)}$. Since $\tilde{\mathcal{H}}_{\mu,j} \cong \mathcal{H}_{\Delta}(\mu_j)_{\mathbb{C}}$ and $\Omega_{\mathbb{C}}^{\otimes \mu_j}$ is a right $\mathcal{H}_{\Delta}(\mu_j)_{\mathbb{C}}$ -module, $\Omega_{\mathbb{C}}^{\otimes \mu_j}$ can be regarded as a right $\tilde{\mathcal{H}}_{\mu,j}$ -module.

For $\mathbf{s} = \{s_1, \dots, s_p\} \in \mathcal{S}_{r,\mu}$, let $\mathbf{a} = (s_1, \dots, s_p) \in (\mathbb{C}^*)^r$ be the r -tuple obtained by juxtaposing the segments in \mathbf{s} . For $1 \leq j \leq p$ let $\mathcal{J}_{\mu,j}$ be the left ideal of $\tilde{\mathcal{H}}_{\mu,j}$ generated by $X_k - a_k$ for $\mu_{[1,j-1]} + 1 \leq k \leq \mu_{[1,j]}$. Let $\iota_j : \mathcal{H}_{\mu,j} \rightarrow \tilde{\mathcal{H}}_{\mu,j} / \mathcal{J}_{\mu,j}$ be the natural $\mathcal{H}_{\mu,j}$ -module isomorphism defined by sending h to \bar{h} . Let

$$\bar{\mathcal{I}}_{\mu,j} = \iota_j(\mathcal{I}_{\mu,j}) = \mathcal{H}_{\mu,j} \bar{y}_{\mu(j)} = \tilde{\mathcal{H}}_{\mu,j} \bar{y}_{\mu(j)}.$$

By 4.3 we have the following corollary.

Corollary 4.4. *Maintain the notation above. There is a $U_{\mathbb{C}}(\mathfrak{gl}_n)$ -module isomorphism*

$$\varphi : (\Omega_{\mathbb{C}}^{\otimes \mu_1} \otimes_{\tilde{\mathcal{H}}_{\mu,1}} \bar{\mathcal{I}}_{\mu,1}) \otimes \cdots \otimes (\Omega_{\mathbb{C}}^{\otimes \mu_p} \otimes_{\tilde{\mathcal{H}}_{\mu,p}} \bar{\mathcal{I}}_{\mu,p}) \rightarrow F(\bar{\mathcal{I}}_{\mu})$$

such that $\varphi(w_1 \otimes \bar{h}_1 \otimes \cdots \otimes w_p \otimes \bar{h}_p) = w_1 \otimes \cdots \otimes w_p \otimes \overline{h_1 \cdots h_p}$ for $w_j \in \Omega_{n,\mathbb{C}}^{\otimes \mu_j}$ and $h_j \in \mathcal{I}_{\mu,j}$ with $1 \leq j \leq p$.

Proof. Combining 3.1, 4.1 with 4.2 yields $F(\bar{\mathcal{I}}_\mu) \cong \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_\mathbb{C}} \bar{\mathcal{J}}_\mu \cong \Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{J}_\mu$ and $\Omega_{\mathbb{C}}^{\otimes \mu_j} \otimes_{\tilde{\mathcal{H}}_{\mu,j}} \bar{\mathcal{I}}_{\mu,j} \cong \Omega_{n,\mathbb{C}}^{\otimes \mu_j} \otimes_{\mathcal{H}_{\mu,j}} \bar{\mathcal{J}}_{\mu,j} \cong \Omega_{n,\mathbb{C}}^{\otimes \mu_j} \mathcal{J}_{\mu,j}$ for $1 \leq j \leq p$. This, together with 4.3, implies the assertion. \square

We now prove that φ is in fact a $U_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ -module isomorphism.

Lemma 4.5. *The map φ is a $U_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ -module homomorphism.*

Proof. Let $u \in U_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ and $w = w_1 \otimes \bar{h}_1 \otimes \cdots \otimes w_p \otimes \bar{h}_p \in (\Omega_{\mathbb{C}}^{\otimes \mu_1} \otimes_{\tilde{\mathcal{H}}_{\mu,1}} \bar{\mathcal{I}}_{\mu,1}) \otimes \cdots \otimes (\Omega_{\mathbb{C}}^{\otimes \mu_p} \otimes_{\tilde{\mathcal{H}}_{\mu,p}} \bar{\mathcal{I}}_{\mu,p})$, where $w_i \in \Omega_{n,\mathbb{C}}^{\otimes \mu_i}$ and $h_i \in \mathcal{I}_{\mu,i}$ for $1 \leq i \leq p$. Assume $\Delta^{(p-1)}(u) = \sum_{(u)} u_1 \otimes \cdots \otimes u_p$, $u_i w_i = \sum_{k_i} w_{i,k_i} g_{i,k_i}$ and $g_{i,k_i} h_i = \sum_{j_i} g_{i,k_i,j_i} X_{j_i}$, where $w_{i,k_i} \in \Omega_{n,\mathbb{C}}^{\otimes \mu_i}$, $g_{i,k_i} \in \mathcal{H}_{\mu,i}$, and $g_{i,k_i,j_i} \in \mathcal{H}_{\mu,i}$, $X_{j_i} \in \tilde{\mathcal{H}}_{\mu,i}$. Then

$$g_{i,k_i}(\iota_i(h_i)) = g_{i,k_i} \bar{h}_i = \sum_{j_i} a_{j_i} \overline{g_{i,k_i,j_i}}.$$

Hence,

$$\begin{aligned} uw &= \sum_{(u)} u_1 w_1 \otimes \bar{h}_1 \otimes \cdots \otimes u_p w_p \otimes \bar{h}_p \\ &= \sum_{(u)} \sum_{k_1, \dots, k_p} w_{1,k_1} \otimes g_{1,k_1} \bar{h}_1 \otimes \cdots \otimes w_{p,k_p} \otimes g_{p,k_p} \bar{h}_p \\ &= \sum_{(u)} \sum_{\substack{k_1, \dots, k_p \\ j_1, \dots, j_p}} a_{j_1} \cdots a_{j_p} w_{1,k_1} \otimes \overline{g_{1,k_1,j_1}} \otimes \cdots \otimes w_{p,k_p} \otimes \overline{g_{p,k_p,j_p}}. \end{aligned}$$

Since $g_{1,k_1} \cdots g_{p,k_p} \bar{h}_1 \cdots \bar{h}_p = \overline{g_{1,k_1} h_1 \cdots g_{p,k_p} h_p} = \sum_{j_1, \dots, j_p} a_{j_1} \cdots a_{j_p} \overline{g_{1,k_1,j_1} \cdots g_{p,k_p,j_p}}$, we conclude that

$$\begin{aligned} \varphi(uw) &= \sum_{(u)} \sum_{\substack{k_1, \dots, k_p \\ j_1, \dots, j_p}} a_{j_1} \cdots a_{j_p} w_{1,k_1} \otimes \cdots \otimes w_{p,k_p} \otimes \overline{g_{1,k_1,j_1} \cdots g_{p,k_p,j_p}} \\ &= \sum_{(u)} \sum_{k_1, \dots, k_p} w_{1,k_1} \otimes \cdots \otimes w_{p,k_p} \otimes g_{1,k_1} \cdots g_{p,k_p} \overline{h_1 \cdots h_p} \\ &= \sum_{(u)} u_1 w_1 \otimes \cdots \otimes u_p w_p \otimes \bar{h}_1 \cdots \bar{h}_p \\ &= u(w_1 \otimes \cdots \otimes w_p \otimes \overline{h_1 \cdots h_p}) \\ &= u\varphi(w). \end{aligned}$$

The proof is completed. \square

We can now describe $F(\bar{\mathcal{I}}_{\varphi(\mathbf{s})})$ as follows.

Proposition 4.6. *Let $\mathbf{s} = \{s_1, \dots, s_p\} \in \mathcal{S}_{r,\mu}$. Then $F(\bar{\mathcal{I}}_\mu) = 0$ for $\mathbf{s} \notin \mathcal{S}_r^{(n)}$ and $F(\bar{\mathcal{I}}_\mu) \cong L(\mathbf{Q}_1) \otimes \cdots \otimes L(\mathbf{Q}_p)$ for $\mathbf{s} \in \mathcal{S}_r^{(n)}$, where $\mathbf{Q}_i = (Q_{i,1}(u), \dots, Q_{i,n}(u))$ with $Q_{i,n}(u) = (1 - a_i v^{-n+1} u)^{\delta_{\mu_i, n}}$ and $\frac{Q_{i,j}(uv^{j-1})}{Q_{i,j+1}(uv^{j+1})} = (1 - a_i u)^{\delta_{j, \mu_i}}$ for $1 \leq i \leq p$ and $1 \leq j \leq n-1$.*

Proof. Since $\bar{\mathcal{I}}_{\mu_i} \cong V_{\mathbf{s}_i}$ for $1 \leq i \leq p$, by 4.4 and 4.5 we conclude that $F(\bar{\mathcal{I}}_\mu) = F_{n,r}(\bar{\mathcal{I}}_\mu) \cong F_{n,\mu_1}(V_{\mathbf{s}_1}) \otimes \cdots \otimes F_{n,\mu_p}(V_{\mathbf{s}_p})$. If $\mathbf{s} \notin \mathcal{S}_r^{(n)}$, then there exist $1 \leq k \leq p$ such that $|\mathbf{s}_k| = \mu_k > n$. By 3.3 we have $F_{n,\mu_k}(V_{\mathbf{s}_k}) = 0$ and hence $F(\bar{\mathcal{I}}_\mu) = 0$. If $\mathbf{s} \in \mathcal{S}_r^{(n)}$, then by 3.4 we have $F_{n,\mu_i}(V_{\mathbf{s}_i}) \cong L(\mathbf{Q}_i)$ for $1 \leq i \leq p$. Consequently, $F(\bar{\mathcal{I}}_\mu) \cong L(\mathbf{Q}_1) \otimes \cdots \otimes L(\mathbf{Q}_p)$. \square

We now turn to studying $F(V_{\mathbf{s}})$ for $\mathbf{s} \in \mathcal{S}_r^{(n)}$. To compute $F(V_{\mathbf{s}})$, we need to generalize [5, 7.2] to the case of $n \leq r$. Recall the notation $\Lambda(n, r)$ defined in (3.1.1). Let $\Lambda^+(n, r) = \Lambda(n, r) \cap \Lambda^+(r)$. For $\lambda \in \mathbb{N}^n$ let $L(\lambda)$ be the irreducible $U_{\mathbb{C}}(\mathfrak{gl}_n)$ -module with highest weight λ . For $1 \leq i \leq n$, let $\mathfrak{k}_i = \zeta_r(K_i)$ and

$$\begin{bmatrix} \mathfrak{k}_i; 0 \\ t \end{bmatrix} = \prod_{s=1}^t \frac{\mathfrak{k}_i v^{-s+1} - \mathfrak{k}_i^{-1} v^{s-1}}{v^s - v^{-s}}.$$

For $\mu \in \mathbb{N}^n$ let $\mathfrak{k}_\mu = \begin{bmatrix} \mathfrak{k}_1; 0 \\ \mu_1 \end{bmatrix} \cdots \begin{bmatrix} \mathfrak{k}_n; 0 \\ \mu_n \end{bmatrix}$. The following result is the generalization of [5, 7.2].

Lemma 4.7. *Let $\mu \in \Lambda^+(r)$. Then $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} E_\mu \neq 0$ if and only if $\mu' \in \Lambda(n, r)$, where μ' is the dual partition of μ . Furthermore if $\mu' \in \Lambda^+(n, r)$, then $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} E_\mu \cong L(\mu')$.*

Proof. We choose N such that $N > \max\{n, r\}$. Let $e = \sum_{\mu \in \Lambda(n, r)} \mathfrak{k}_\mu \in \mathcal{S}(N, r)_{\mathbb{C}}$. It is well known that for $\mu \in \Lambda^+(N, r)$, $eL(\mu) \neq 0$ if and only if $\mu \in \Lambda(n, r)$ (cf. [9, 6.5(f)]). Furthermore by [7, 4.3.3] and [5, 7.2] we have $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} E_\mu \cong e(\Omega_{N,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} E_\mu) \cong e(L(\mu'))$. Thus $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} E_\mu \neq 0$ if and only if $\mu' \in \Lambda(n, r)$. If $\mu' \in \Lambda^+(n, r)$, then $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} E_\mu \cong e(L(\mu')) \cong L(\mu')$. \square

In the case of $n > r$, the Drinfeld polynomials associated with $F(V_{\mathbf{s}})$ were calculated for $\mathbf{s} \in \mathcal{S}_r^{(n)}$ in [5, 7.6] and [7, 4.4.2]. We are now prepared to use 4.6 and 4.7 to generalize [loc. cit.] to the case of $n \leq r$ in 4.9.

Let $\mathcal{Q}(n)_r = \{\mathbf{Q} \in \mathcal{Q}(n) \mid \sum_{1 \leq i \leq n} \deg Q_i(u) = r\}$. For $\mathbf{s} = \{\mathbf{s}_1, \dots, \mathbf{s}_p\} \in \mathcal{S}_r^{(n)}$ with

$$\mathbf{s}_i = (a_i v^{-\mu_i+1}, a_i v^{-\mu_i+3}, \dots, a_i v^{\mu_i-1}) \in (\mathbb{C}^*)^{\mu_i},$$

define $\mathbf{Q}_{\mathbf{s}} = (Q_1(u), \dots, Q_n(u))$ by setting $Q_n(u) = \prod_{\substack{1 \leq i \leq p \\ \mu_i = n}} (1 - a_i u v^{-n+1})$ and

$$Q_i(u) = P_i(u v^{-i+1}) P_{i+1}(u v^{-i+2}) \cdots P_{n-1}(u v^{n-2i}) Q_n(u v^{2(n-i)})$$

for $1 \leq i \leq n-1$, where

$$P_i(u) = \prod_{\substack{1 \leq j \leq p \\ \mu_j = i}} (1 - a_j u).$$

Then

$$\sum_{1 \leq i \leq n} \deg Q_i(u) = n \deg Q_n(u) + \sum_{1 \leq i \leq n-1} i \deg P_i(u) = \sum_{1 \leq i \leq p} \mu_i = r.$$

So $\mathbf{Q}_{\mathbf{s}} \in \mathcal{Q}(n)_r$. Consequently, we obtain a map $\partial_{n,r} : \mathcal{S}_r^{(n)} \rightarrow \mathcal{Q}(n)_r$ defined by sending \mathbf{s} to $\mathbf{Q}_{\mathbf{s}}$.

Lemma 4.8. *The map $\partial_{n,r} : \mathcal{S}_r^{(n)} \rightarrow \mathcal{Q}(n)_r$ is bijective.*

Proof. It is clear that $\partial_{n,r}$ is injective. Let $\mathbf{Q} = (Q_1(u), \dots, Q_n(u)) \in \mathcal{Q}(n)_r$ and let $\lambda \in \Lambda(n, r)$, with $\lambda_i = \deg Q_i(u)$. For $1 \leq j \leq n-1$ let

$$P_j(u) = \frac{Q_j(uv^{j-1})}{Q_{j+1}(uv^{j+1})}$$

and $\nu_j = \deg P_j(u) = \lambda_j - \lambda_{j+1}$. We write, for $1 \leq i \leq n-1$,

$$P_i(u) = (1 - a_{\nu_1 + \dots + \nu_{i-1} + 1}u)(1 - a_{\nu_1 + \dots + \nu_{i-1} + 2}u) \cdots (1 - a_{\nu_1 + \dots + \nu_{i-1} + \nu_i}u),$$

and $Q_n(u) = (1 - b_1u) \cdots (1 - b_{\lambda_n}u)$. Let $p' = \sum_{1 \leq i \leq n-1} \nu_i$ and $p = p' + \lambda_n$. Let $\mathbf{s} = \{\mathbf{s}_1, \dots, \mathbf{s}_p\}$, where

$$\mathbf{s}_i = \begin{cases} (a_i v^{-\mu_i+1}, a_i v^{-\mu_i+3}, \dots, a_i v^{\mu_i-1}) & \text{for } 1 \leq i \leq p' \\ (b_{i-p'}, b_{i-p'} v^2, \dots, b_{i-p'} v^{2(n-1)}) & \text{for } p' + 1 \leq i \leq p \end{cases}$$

and $(\mu_1, \dots, \mu_{p'}) = (1^{\nu_1}, \dots, (n-1)^{\nu_{n-1}})$. Since

$$\sum_{1 \leq i \leq p} |\mathbf{s}_i| = \sum_{1 \leq j \leq p'} \mu_j + n\lambda_n = \sum_{1 \leq i \leq n-1} i\nu_i + n\lambda_n = \sum_{1 \leq i \leq n} \lambda_i = r,$$

we have $\mathbf{s} \in \mathcal{S}_r^{(n)}$. It is easy to see that $\partial_{n,r}(\mathbf{s}) = \mathbf{Q}$. Thus $\partial_{n,r}$ is surjective. \square

Theorem 4.9. *For $\mathbf{s} = \{\mathbf{s}_1, \dots, \mathbf{s}_p\} \in \mathcal{S}_r^{(n)}$ with $\mathbf{s}_i = (a_i v^{-\mu_i+1}, a_i v^{-\mu_i+3}, \dots, a_i v^{\mu_i-1})$, we have $F(V_{\mathbf{s}}) \cong L(\mathbf{Q}_{\mathbf{s}})$, where $\mathbf{Q}_{\mathbf{s}} = \partial_{n,r}(\mathbf{s})$. In particular we have $F(V_{\mathbf{s}})|_{U_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)} \cong \bar{L}(\mathbf{P})$, where*

$$P_i(u) = \prod_{\substack{1 \leq j \leq p \\ \mu_j = i}} (1 - a_j u).$$

for $1 \leq i \leq n-1$.

Proof. Let $W = F(\bar{\mathcal{I}}_{\mu})$. By 4.6 we have $W \cong L(\mathbf{Q}_1) \otimes \cdots \otimes L(\mathbf{Q}_p)$, where $\mathbf{Q}_i = (Q_{i,1}(u), \dots, Q_{i,n}(u))$ with $Q_{i,n}(u) = (1 - a_i v^{-n+1}u)^{\delta_{\mu_i, n}}$ and

$$P_{i,j}(u) := \frac{Q_{i,j}(uv^{j-1})}{Q_{i,j+1}(uv^{j+1})} = (1 - a_i u)^{\delta_{j, \mu_i}}$$

for $1 \leq i \leq p$ and $1 \leq j \leq n-1$. We will identify W with $L(\mathbf{Q}_1) \otimes \cdots \otimes L(\mathbf{Q}_p)$. Let $w = w_1 \otimes \cdots \otimes w_p \in W$, where w_i is the pseudo-highest weight vector in $L(\mathbf{Q}_i)$. Then by [5, 6.3] and [8, 4.1] we conclude that w is the pseudo-highest weight vector in W such that $\mathbf{k}_i w = v^{\lambda_i} w$ and $\mathcal{Q}_i^{\pm}(u)w = Q_i^{\pm}(u)w$ for $1 \leq i \leq n$, where $\lambda_i = \deg Q_i^+(u)$,

$$Q_n^{\pm}(u) = \prod_{1 \leq i \leq p} Q_{i,n}^{\pm}(u) = \prod_{1 \leq i \leq p} (1 - (a_i u)^{\pm 1} v^{\pm(-n+1)})^{\delta_{\mu_i, n}} = \prod_{\substack{1 \leq i \leq p \\ \mu_i = n}} (1 - (a_i u)^{\pm 1} v^{\pm(-n+1)})$$

and

$$P_j^\pm(u) := \frac{Q_j^\pm(v^{j-1}u)}{Q_{j+1}^\pm(v^{j+1}u)} = \prod_{1 \leq i \leq p} P_{i,j}^\pm(u) = \prod_{1 \leq i \leq p} (1 - (a_i u)^{\pm 1})^{\delta_{j,\mu_i}} = \prod_{\substack{1 \leq i \leq p \\ \mu_i = j}} (1 - (a_i u)^{\pm 1})$$

for $1 \leq j \leq n-1$. By definition we have $\mathbf{Q}_s = (Q_1^+(u), \dots, Q_n^+(u))$. Since $\lambda_j = \deg Q_j^+(u) = \lambda_n + \sum_{j \leq s \leq n-1} \deg P_s^+(u) = |\{1 \leq i \leq p \mid \mu_i \geq j\}|$ for $1 \leq j \leq n$, we have $\lambda = (\lambda_1, \dots, \lambda_n) = \mu'$.

Let $L = F(V_s)$. Since V_s is a semisimple $\mathcal{H}(r)_\mathbb{C}$ -module, by 3.1 and 4.7 we have $[L : L(\lambda)] = [L : \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_\mathbb{C}} E_\mu] = [\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_\mathbb{C}} V_s : \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_\mathbb{C}} E_\mu] = [V_s : E_\mu] = 1$. Thus

$$(4.9.1) \quad \dim L_\lambda = 1.$$

Since V_s is the irreducible subquotient of \tilde{L}_μ , there is a surjective $U_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ -module homomorphism $f : M \rightarrow L$, where M is a certain submodule of W . Since $1 = \dim L_\lambda \leq \dim M_\lambda \leq \dim W_\lambda = 1$, we conclude that $\dim M_\lambda = \dim W_\lambda = 1$. Hence $M_\lambda = W_\lambda = \text{span}\{w\}$ and $L_\lambda = \text{span}\{f(w)\}$. By (4.9.1) we have $f(w) \neq 0$. Since f is a $U_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ -module homomorphism, $f(w)$ is the pseudo-highest weight vector in L such that $\mathbf{k}_i f(w) = f(\mathbf{k}_i w) = v^{\lambda_i} f(w)$ and $Q_i^\pm(u) f(w) = f(Q_i^\pm(u) w) = Q_i^\pm(u) f(w)$ for $1 \leq i \leq n$. This implies that L is the irreducible quotient module of $M(\mathbf{Q}_s)$ and hence $L \cong L(\mathbf{Q}_s)$. \square

Combining 3.3, 4.8 with 4.9 yields the following classification theorem of irreducible $\mathcal{S}_\Delta(n, r)_\mathbb{C}$ -modules, which was proved in [7, 4.6.8] using a different approach.

Corollary 4.10. *The set $\{L(\mathbf{Q}) \mid \mathbf{Q} \in \mathcal{Q}(n)_r\}$ is a complete set of nonisomorphic finite dimensional irreducible $\mathcal{S}_\Delta(n, r)_\mathbb{C}$ -modules.*

Finally we will use 4.9 to generalize [9, (6.5f)] to the affine case in 4.11. Assume $N \geq n$. Let $e = \sum_{\lambda \in \Lambda(n, r)} \mathbf{k}_\lambda \in \mathcal{S}_\Delta(N, r)_\mathbb{C}$. Then $e\mathcal{S}_\Delta(N, r)_\mathbb{C}e \cong \mathcal{S}_\Delta(n, r)_\mathbb{C}$. Consequently, the categories $e\mathcal{S}_\Delta(N, r)_\mathbb{C}e\text{-mod}$ and $\mathcal{S}_\Delta(n, r)_\mathbb{C}\text{-mod}$ may be identified. With this identification, we define a functor

$$(4.10.1) \quad \mathbf{G} = \mathbf{G}_{N, n, r} : \mathcal{S}_\Delta(N, r)_\mathbb{C}\text{-mod} \longrightarrow \mathcal{S}_\Delta(n, r)_\mathbb{C}\text{-mod}, \quad V \longmapsto eV.$$

Then by definition we have $\mathbf{G}_{N, n, r} \circ \mathbf{F}_{N, r} = \mathbf{F}_{n, r}$. For $\mathbf{Q} = (Q_1(u), \dots, Q_n(u)) \in \mathcal{Q}(n)_r$ let $\tilde{\mathbf{Q}} = (Q_1(u), \dots, Q_n(u), 1, \dots, 1) \in \mathcal{Q}(N)_r$. Let $\tilde{\mathcal{Q}}(n)_r = \{\tilde{\mathbf{Q}} \mid \mathbf{Q} \in \mathcal{Q}(n)_r\} \subseteq \mathcal{Q}(N)_r$. Clearly, by definition, we have

$$(4.10.2) \quad \partial_{N, r}(\mathbf{s}) = \widetilde{\partial_{n, r}(\mathbf{s})}.$$

for $\mathbf{s} \in \mathcal{S}_r^{(n)}$.

Theorem 4.11. *Assume $N \geq n$. Then $\mathbf{G}(L(\tilde{\mathbf{Q}})) \cong L(\mathbf{Q})$ for $\mathbf{Q} \in \mathcal{Q}(n)_r$. In particular we have $\dim L(\tilde{\mathbf{Q}})_\alpha = \dim L(\mathbf{Q})_\alpha$ for $\alpha \in \Lambda(n, r)$. Furthermore, for $\mathbf{Q}' \in \mathcal{Q}(N)_r$, $\mathbf{G}(L(\mathbf{Q}')) \neq 0$ if and only if $\mathbf{Q}' \in \tilde{\mathcal{Q}}(n)_r$.*

Proof. If $\mathbf{Q} \in \mathcal{Q}(n)_r$ then by 4.8 we conclude that there exist $\mathbf{s} \in \mathcal{S}_r^{(n)}$ such that $\mathbf{Q} = \partial_{n,r}(\mathbf{s})$. By 4.9 and (4.10.2) we have $L(\tilde{\mathbf{Q}}) \cong \mathcal{T}_\Delta(N, r) \otimes_{\mathcal{H}_\Delta(r)_\mathbb{C}} V_{\mathbf{s}}$. So by [7, 4.3.3] and 4.9 we have $G(L(\tilde{\mathbf{Q}})) \cong (e\mathcal{T}_\Delta(N, r)) \otimes_{\mathcal{H}_\Delta(r)_\mathbb{C}} V_{\mathbf{s}} \cong \mathcal{T}_\Delta(n, r) \otimes_{\mathcal{H}_\Delta(r)_\mathbb{C}} V_{\mathbf{s}} \cong L(\mathbf{Q})$. By [9, 6.2(g)], the set $\{G(L(\mathbf{Q}')) \neq 0 \mid \mathbf{Q}' \in \mathcal{Q}(N)_r\}$ forms a complete set of non-isomorphic irreducible $\mathcal{S}_\Delta(n, r)_\mathbb{C}$ -modules. This together with 4.10 implies that $\{G(L(\mathbf{Q}')) \neq 0 \mid \mathbf{Q}' \in \mathcal{Q}(N)_r\} = \{G(L(\tilde{\mathbf{Q}})) \mid \mathbf{Q} \in \mathcal{Q}(n)_r\}$. Consequently, $G(L(\mathbf{Q}')) \neq 0$ if and only if $\mathbf{Q}' \in \tilde{\mathcal{Q}}(n)_r$. \square

REFERENCES

- [1] J. Beck, *Braid group action and quantum affine algebras*, Comm. Math. Phys. **165** (1994), 655–568.
- [2] V. Chari and A. Pressley, *Quantum affine algebras*, Comm. Math. Phys. **142** (1991), 261–283.
- [3] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, 1994.
- [4] V. Chari and A. Pressley, *Quantum affine algebras and their representations*, Representations of groups (Banff, AB, 1994), 59–78, CMS Conf. Proc., 16, Amer. Math. Soc., Providence, RI, 1995.
- [5] V. Chari and A. Pressley, *Quantum affine algebras and affine Hecke algebras*, Pacific J. Math. **174** (1996), 295–326.
- [6] V. Chari and A. Pressley, *Quantum affine algebras at roots of unity*, Represent. Theory **1** (1997), 280–328.
- [7] B.B. Deng, J. Du and Q. Fu, *A Double Hall Algebra Approach to Affine Quantum Schur–Weyl Theory*, LMS Lecture Notes Series (to appear).
- [8] E. Frenkel and E. Mukhin, *The Hopf algebra Rep $U_q(\widehat{\mathfrak{gl}}_\infty)$* , Sel. math., New Ser. **8** (2002), 537–635.
- [9] J. A. Green, *Polynomial Representations of GL_n* , 2nd ed., with an appendix on Schensted correspondence and Littelmann paths by K. Erdmann, J. A. Green and M. Schocker, Lecture Notes in Mathematics, no. 830, Springer-Verlag, Berlin, 2007.
- [10] M. Jimbo, *A q -analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang–Baxter equation*, Letters in Math. Physics **11** (1986), 247–252.
- [11] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke Algebras*, Invent. Math. **53** (1979), 165–184.
- [12] J. D. Rogawski, *On modules over the Hecke algebra of a p -adic group*, Invent. Math. **79** (1985), 443–465.
- [13] A. V. Zelevinsky, *Induced representations of reductive p -adic groups II. On irreducible representations of GL_n* , Ann. Sci. Ec. Norm. Sup. 4^e Sér. **13** (1980), 165–210.

DEPARTMENT OF MATHEMATICS, TONGJI UNIVERSITY, SHANGHAI, 200092, CHINA.

E-mail address: q.fu@hotmail.com